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On the stabilization of periodic orbits for discrete time chaotic systems

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Abstract

In this Letter we consider the stabilization problem of unstable periodic orbits of discrete time chaotic systems. We propose a novel and simple periodic delayed feedback law and present some stability results. These results show that all hyperbolic periodic orbits as well as some non-hyperbolic periodic orbits can be stabilized with the proposed method. The stability proofs also give the possible feedback gains which achieve stabilization. We will also present some simulation results.
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1. Introduction

After the seminal work of [1], where the term "controlling chaos" was introduced, the interest in the study of various aspects of chaotic systems has received great interest among scientists from various fields due to their numerous potential applications [2]. Among such aspects, the problem of controlling chaos as mentioned in [1] is an important subject. As in classical control theory, various control problems can be defined for chaotic systems as well. Among such problems, an important one, which was investigated in [1], is to obtain simple schemes which stabilize some unstable periodic orbits. As is well known, chaotic systems usually have infinitely many periodic orbits embedded in their attractors, most of which are unstable [3]. As was shown in [1], by using appropriate inputs, some of these orbits may be stabilized. Following the work of [1], various schemes have been proposed for this as well as other control problems for chaotic systems, [2,4]. Among these methods, the delayed feedback scheme (DFC), first proposed by Pyragas in [5], has gained attention due to its simplicity. In this scheme, the required input for stabilization is the difference between the current and one period de-

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layed states multiplied by a gain, and the problem is to find an appropriate gain which stabilizes a given unstable periodic orbit. DFC is then successfully applied to many systems including lasers, electronic oscillators, chemical systems, etc., and for details see, e.g., [2,6], and the references therein.

Despite its simplicity, a detailed stability analysis of DFC seems to be very difficult, see, e.g., [6–8]. These results show that classical DFC has some inherent limitations, i.e., it cannot stabilize certain unstable periodic orbits, see, e.g., [7,9,10]. To overcome these limitations, some modifications of DFC have been proposed, see, e.g., [10–16]. Among these, the periodic feedback scheme proposed in [15] seems to be promising due to its simple structure. This method eliminates most of the limitations of DFC for the period 1 case, and various extensions to higher period cases are possible. In [15] such an extension was given, but as stated in [6], the related stability result is not clear. In this Letter we will propose another extension of such a periodic feedback scheme, which is different than the one proposed in [15]. We will show that the resulting feedback system achieves the stabilization of a given periodic orbit under a very mild condition. This condition is related to the hyperbolicity of the related periodic orbit and we will prove that all hyperbolic periodic orbits as well as some non-hyperbolic orbits can be stabilized with the proposed scheme. This point is interesting since recently it was shown that the method of [1] may fail to stabilize some non-hyperbolic periodic orbits, [17].

This Letter is organized as follows. In the next section we will formally state the problem under investigation and present some notation which will be used in the sequel. In Section 3 we will propose our feedback scheme to solve the proposed problem. In Section 4 we will provide some stability results. In the next section we will present a simple implementation of the proposed scheme by utilizing its local nature. After some simulation results we will give some concluding remarks.

2. Problem statement

Let us consider the following discrete-time system

$$x(k+1) = f\big(x(k)\big), \tag{1}$$

where $k = 1, 2, \ldots$ is the discrete time index, $x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is an appropriate function, which is assumed to be differentiable wherever required. We assume that the system given by (1) possesses a T periodic orbit characterized by the set

$$\Sigma_T = \{x_1^*, x_2^*, \ldots, x_T^*\}, \tag{2}$$

i.e., for $x(1) = x_1^*$, the iterates of (1) yields $x(2) = x_2^*, \ldots, x(T) = x_T^*$, $x(k) = x(k-T)$ for $k > T$.

Let $x(\cdot)$ be a solution of (1). To characterize the convergence of $x(\cdot)$ to Σ_T, we need a distance measure, which is defined as follows. For x_i^*, we will use circular notation, i.e., $x_i^* = x_j^*$ for $i = j \pmod{(T)}$. Let us define the following indices ($j = 1, \ldots, T$):

$$d_k(j) = \sqrt{\sum_{i=0}^{T-1} \big\| x(k+i) - x_{i+j}^* \big\|^2}, \tag{3}$$

where $\|\cdot\|$ denotes any norm in $\mathbf{R}^n$. Without loss of generality, we will use standard Euclidean norm in the sequel. We then define the following distance measure

$$d\big(x(k), \Sigma_T\big) = \min\big\{d_k(1), \ldots, d_k(T)\big\}. \tag{4}$$

Clearly, if $x(1) \in \Sigma_T$, then $d(x(k), \Sigma_T) = 0$, $\forall k$. Conversely if $d(x(k), \Sigma_T) = 0$ for some k_0, then it remains 0 and $x(k) \in \Sigma_T$, for $k \geqslant k_0$. We will use $d(x(k), \Sigma_T)$ as a measure of convergence to the periodic solution given by Σ_T.

Let $x(\cdot)$ be a solution of (1) starting with $x(1) = x_1$. We say that Σ_T is (locally) asymptotically stable if there exists an $\varepsilon > 0$ such that for any $x(1) \in \mathbf{R}^n$ for which $d(x(1), \Sigma_T) < \varepsilon$ holds, we have $\lim_{k\to\infty} d(x(k), \Sigma_T) = 0$. Moreover, if this decay is exponential, i.e., the following holds for some $M \geqslant 1$ and $0 < \rho < 1$ ($k > 1$):

$$d\big(x(k), \Sigma_T\big) \leqslant M\rho^k d\big(x(1), \Sigma_T\big), \tag{5}$$

then we say that Σ_T is (locally) exponentially stable.

To stabilize the periodic orbits of (1), let us apply the following control law:

$$x(k+1) = f\big(x(k)\big) + u(k), \tag{6}$$

where $u(\cdot)$ is the control input. In classical DFC, the following feedback law is used ($k > T$):

$$u(k) = K\big(x(k) - x(k-T)\big), \tag{7}$$

where $K \in \mathbf{R}^{n\times n}$ is a constant gain matrix to be determined. It is known that the scheme given above has certain inherent limitations, see, e.g., [7]. For example, assume that $n = 1$ and let $\Sigma_1 = \{x_1^*\}$ be a period 1 orbit of (1) and set $a_1 = f'(x_1^*)$, where a prime denotes the derivative. It can be shown that Σ_1 can be stabilized with this scheme if $-3 < a_1 < 1$ and cannot be stabilized if $a_1 > 1$, see [7]. For Σ_T, let us set $a_i = f'(x_i^*)$. It can be shown that Σ_T cannot be stabilized with this scheme if $\prod_{i=1}^{T} a_i > 1$, see, e.g., [7,8], and a similar condition can be generalized to the case $n > 1$ [10]. A set of necessary and sufficient conditions to guarantee exponential stabilization for $n = 1$ can be found in [8].

3. Single period delayed feedback scheme

To overcome the limitations of DFC scheme, various modifications have been proposed, see [10–16]. One of these schemes is the so-called periodic, or oscillating feedback, see [15]. For period 1 case, the corresponding feedback law is given by:

$$u(k) = \epsilon(k)\big(x(k) - x(k-1)\big), \tag{8}$$

where $\epsilon(k)$ is given as:

$$\epsilon(k) = \begin{cases} K, & k \ \big(\text{mod}\ (2)\big) = 0, \\ 0, & k \ \big(\text{mod}\ (2)\big) \neq 0, \end{cases} \tag{9}$$

where $K \in \mathbf{R}^{n\times n}$ is a constant gain matrix to be determined. It is well known that this scheme eliminates the limitations of classical DFC, for the case $m = 1$, see, e.g., [15].

The idea given above can be generalized to the case $T = m > 1$. One particular generalization is given in [15]. However, as noted in [6], the stability analysis given in [15] is not clear. In the sequel, we will provide a different generalization along with a simple stability analysis.

As a generalization of the control law given by (8), (9) for the case $T = m > 1$, we propose the following control law:

$$u(k) = \epsilon(k)\big(x(k) - x(k-m)\big), \tag{10}$$

where $\epsilon(k)$ is given as:

$$\epsilon(k) = \begin{cases} K, & k \ \big(\text{mod}(m+1)\big) = 0, \\ 0, & k \ \big(\text{mod}(m+1)\big) \neq 0. \end{cases} \tag{11}$$

Clearly, for $m = 1$, both (10) and (11) reduces to (8), and (9), respectively. For the sake of clarity, we will call the scheme given by (10) and (11) as single period delayed feedback scheme (SPDFC).

4. Stability analysis

To motivate our analysis, let us consider the case $m = 2$ first. Let the period 2 orbit be given as $\Sigma_2 = \{x_1^*, x_2^*\}$; hence we have

$$x_2^* = f\big(x_1^*\big), \qquad x_1^* = f\big(x_2^*\big). \tag{12}$$

Let us define the error $e(\cdot)$ as:

$$e(i) = x(i) - x_i^*, \tag{13}$$

and let us define the Jacobian matrices J_i evaluated at periodic points as:

$$J_i = \frac{\partial f}{\partial x}\bigg|_{x=x_i^*}, \quad i = 1, 2, \ldots, m, \tag{14}$$

where here and in the sequel we will use circular notation for x_i^*, and J_i, i.e.

$$x_i^* = x_j^*, \qquad J_i = J_j, \quad i = j \ \big(\text{mod}\ (m)\big). \tag{15}$$

By using linear approximation, (6), (10)–(14), we obtain:

$$e(2) = x(2) - x_2^* = f\big(x(1)\big) - f\big(x_1^*\big) = J_1 e(1), \tag{16}$$

$$\begin{aligned} e(3) &= x(3) - x_3^* = x(3) - x_1^* \\ &= f\big(x(2)\big) - f\big(x_2^*\big) = J_2 e(2), \end{aligned} \tag{17}$$

$$\begin{aligned} e(4) &= x(4) - x_4^* = x(4) - x_2^* \\ &= f\big(x(3)\big) - f\big(x_1^*\big) + K\big(x(3) - x(1)\big) \\ &= (J_1 + K)e(3) - Ke(1). \end{aligned} \tag{18}$$

Hence, by using (16), (17) in (18) we obtain:

$$\begin{aligned} e(4) &= \big((J_1 + K)J_2J_1 - K\big)e(1) \\ &= \big(J_1J_2J_1 + K(J_2J_1 - I)\big)e(1). \end{aligned} \tag{19}$$

Proceeding similarly, we obtain:

$$\begin{aligned} e(5) &= x(5) - x_5^* = x(5) - x_1^* \\ &= f\big(x(4)\big) - f\big(x_2^*\big) = J_2 e(4), \end{aligned} \tag{20}$$

$$\begin{aligned} e(6) &= x(6) - x_6^* = x(6) - x_2^* \\ &= f\big(x(5)\big) - f\big(x_1^*\big) = J_1 e(5), \end{aligned} \tag{21}$$

$$\begin{aligned} e(7) &= x(7) - x_7^* = x(7) - x_1^* \\ &= f\bigl(x(6)\bigr) - f\bigl(x_2^*\bigr) + K\bigl(x(6) - x(4)\bigr) \\ &= (J_2 + K)e(6) - Ke(4). \end{aligned} \tag{22}$$

Hence, by using (20), (21) in (22) we obtain:

$$\begin{aligned} e(7) &= \bigl((J_2 + K)J_1J_2 - K\bigr)e(4) \\ &= \bigl(J_2J_1J_2 + K(J_1J_2 - I)\bigr)e(4). \end{aligned} \tag{23}$$

Let us define the matrices P_1 and P_2 as follows:

$$\begin{aligned} P_1 &= \bigl(J_1J_2J_1 + K(J_2J_1 - I)\bigr), \\ P_2 &= \bigl(J_2J_1J_2 + K(J_1J_2 - I)\bigr). \end{aligned} \tag{24}$$

Now by using (24) and (19) in (23), we obtain:

$$e(7) = P_2P_1e(1). \tag{25}$$

Repeating the same argument, we easily obtain:

$$e\bigl[k(m+1)+1\bigr] = P_k e\bigl[(k-1)(m+1)+1\bigr], \quad k = 1, 2, \ldots, \tag{26}$$

where we use the circular notation for P_k, e.g.,

$$P_k = P_l, \quad k = l \;\bigl(\mathrm{mod}\ (m)\bigr). \tag{27}$$

By using (26) and (27), clearly we obtain:

$$e\bigl(2j(m+1)+1\bigr) = (P_2P_1)^j e(1). \tag{28}$$

Clearly we will have $e(k) \to 0$ as $k \to \infty$ if and only if the matrix P_2P_1 is stable (i.e., all of its eigenvalues are strictly inside the unit circle).

Remark 1. On the other hand, if we start (26) from $k = 2$, we obtain

$$e\bigl[(2j+1)(m+1)+1\bigr] = (P_1P_2)^j e(4). \tag{29}$$

Clearly we will have $e(k) \to 0$ as $k \to \infty$ if and only if the matrix P_1P_2 is stable. At a first glance this might seem to be inconsistent with our previous stability statement. But note that the matrices P_2P_1 and P_1P_2 share the same eigenvalues, hence they have the same stability properties, see, e.g., [3, Lemma A.2, p. 558].

Now the question is whether we can make the matrix P_2P_1 (and hence P_1P_2) stable by appropriate choice of the gain matrix K. Next, we will show that this is possible under mild conditions. Note that a matrix A is stable when $\|A\| < 1$, where $\|\cdot\|$ is any operator norm. Now consider (24). Now, if $(J_2J_1 - I)$ is invertible, then by choosing

$$K = K_1 = -J_1J_2J_1(J_2J_1 - I)^{-1}, \tag{30}$$

we will have $P_1 = 0$. Similarly, if $(J_1J_2 - I)$ is invertible, then by choosing

$$K = K_2 = -J_2J_1J_2(J_1J_2 - I)^{-1}, \tag{31}$$

we will have $P_2 = 0$. On the other hand, we have $\|P_2P_1\| \leqslant \|P_1\|\|P_2\|$. Hence from (24), (30), and (31) we see that when $K = K_1$ or $K = K_2$, the matrix P_2P_1 (and hence P_1P_2) will be stable, hence Σ_2 will be stabilized with this choice. Note that by continuity, if K is sufficiently close to K_1 or K_2, this property will still hold, see Remark 2 below. Since the eigenvalues of J_2J_1 and J_1J_2 are the same, the matrices $(J_2J_1 - I)$ and hence $(J_1J_2 - I)$ are invertible if and only if the matrix J_2J_1 (and hence J_1J_2) does not have and eigenvalue $\lambda = 1$. We can summarize these results in the following theorem:

Theorem 1. *Let $\Sigma_2 = \{x_1^*, x_2^*\}$ be a period 2 orbit of (1) and let us define the related Jacobians J_1, J_2 as given in (14). Consider the system given by (6), (10), (11). There exists a gain matrix K such that Σ_2 is locally exponentially stable if and only if the matrix J_2J_1 (and hence J_1J_2) does not have an eigenvalue $\lambda = 1$.*

Proof. Note that the local exponential stability is equivalent to the stability of the linearized system, see, e.g., [20]. The sufficiency of the stated condition is obvious from the analysis given above; simply by choosing $K = K_1$ or $K = K_2$, we achieve stability of the linearized system, hence for the original system Σ_2 is locally exponentially stable. Conversely, assume that J_2J_1 has an eigenvalue $\lambda = 1$, and let ϕ be the corresponding eigenvector, i.e., we have $J_2J_1\phi = \phi$. By using (24), we obtain

$$\begin{aligned} P_2P_1\phi &= P_2\bigl[J_1J_2J_1 + K(J_2J_1 - I)\bigr]\phi = P_2J_1\phi \\ &= \bigl(J_2J_1J_2 + K(J_1J_2 - I)\bigr)J_1\phi \\ &= \phi + KJ_1(J_2J_1 - I)\phi = \phi. \end{aligned} \tag{32}$$

Hence P_2P_1 has an eigenvalue $\lambda = 1$, therefore it cannot be stable. Therefore, Σ_2 cannot be locally exponentially stable. □

Remark 2. Let us assume that the conditions stated in the Theorem 1 holds. By choosing $K = K_1$ or

$K = K_2$, we achieve stabilization of Σ_2 and that $P_2P_1 = P_1P_2 = 0$. Let Δ_i be a sufficiently small matrix and choose $K = K_i + \Delta_i$. Then we have $P_i = \Delta_i(J_{i+1}J_{i+2} - I)$ (note that we have circular notation, see (15)), $P_{i+1} = C_{i1} + \Delta_i C_{i2}$, where C_{i1} and C_{i2} are appropriate matrices depending on the Jacobian matrices J_i. Hence we will have

$$\|P_2P_1\| \leqslant \|\Delta_i\|\|J_{i+1}J_{i+2} - I\| \times \left(\|C_{i1}\| + \|\Delta_i\|\|C_{i2}\|\right). \tag{33}$$

Clearly as $\Delta_i \to 0$, the upper bound in (33) will approach to 0. Hence, there exist bounds $\bar{\Delta}_1, \bar{\Delta}_2$ such that when $\|\Delta_i\| < \bar{\Delta}_i$, we have $\|P_2P_1\| < 1$, hence stabilization occurs. Therefore for any gain matrix K satisfying $\|K - K_i\| < \bar{\Delta}_i$, $i = 1, 2$, stabilization occurs.

Now let us consider the general case $T = m$. Let a period m solution of (1) be given as $\Sigma_m = \{x_1^*, x_2^*, \ldots, x_m^*\}$. Let us define the related Jacobian matrices as given in (14). By using (6), (10), (11), the fact that $x_{i+1}^* = f(x_i^*)$ for $i = 1, 2, \ldots, m$, and by repeating the analysis between (16)–(26), similar to (26), we obtain:

$$e\left[k(m+1)+1\right] = P_k e\left[(k-1)(m+1)+1\right], \quad k = 1, 2, \ldots, \tag{34}$$

where P_k is given by

$$P_k = (J_k + K)J_{k+m-1}J_{k+m-2}\cdots J_{k+1}J_k - K, \tag{35}$$

where we use circular notation for P_k and J_k, see (15), (27). By using (34) repeatedly, we obtain:

$$e\left[mj(m+1)+1\right] = P^j e(1), \quad j = 1, 2, \ldots, \tag{36}$$

where P is given as:

$$P = P_m P_{m-1}\cdots P_2 P_1. \tag{37}$$

Clearly we will have $e(k) \to 0$ as $k \to \infty$ if and only if the matrix P is stable. By starting (34) from various initial points k, we may obtain various circular multiplications of P_i, which may seem to yield different error equations, cf. (28), (29). To show that stability is preserved among such error equations, let us formally define the set σ as follows:

$$\sigma = \{1, 2, \ldots, m-1, m\}, \tag{38}$$

and let $\sigma(j)$ be any j circular permutation of the elements of σ, defined as follows:

$$\sigma(j) = \{j, j+1, \ldots, m+j-2, m+j-1\}, \tag{39}$$

where we have circular notation, e.g., $j = l \pmod{(m)}$. So, we have $\sigma(1) = \sigma$. Accordingly, let us define the matrix $P_{\sigma(j)}$ as follows:

$$P_{\sigma(j)} = P_{m+j-1}P_{m+j-2}\cdots P_{j+1}P_j, \quad j = 1, 2, \ldots, m. \tag{40}$$

Therefore, P given by (37) is also given as $P = P_{\sigma(1)}$. Hence, if we start (34) with $k = i$, then we obtain

$$e\left[(m+i-1)j(m+1)+1\right] = P_{\sigma(i)}^j e\left[(i-1)(m+1)+1\right], \quad j = 1, 2, \ldots. \tag{41}$$

Clearly we will have $e(k) \to 0$ as $k \to \infty$ if and only if the matrix $P_{\sigma(i)}$ is stable. This is not in contradiction with our previous statement, since all matrices $P_{\sigma(i)}$ are circular multiplications of matrices P_i, their eigenvalues are the same, hence they all have the same stability properties, see Remark 1, and see, e.g., [3, Lemma A.2, p. 558].

Similar to (40), let us define the following multiple of Jacobians:

$$J_{\sigma(j)} = J_{m+j-1}J_{m+j-2}\cdots J_{j+1}J_j, \quad j = 1, 2, \ldots, m. \tag{42}$$

By using (42) in (35) we obtain:

$$P_j = J_j J_{\sigma(j)} + K(J_{\sigma(j)} - I), \quad j = 1, 2, \ldots, m. \tag{43}$$

Hence, if $J_{\sigma(j)} - I$ is invertible, then by choosing:

$$K = K_j = -J_j J_{\sigma(j)}(J_{\sigma(j)} - I)^{-1}, \quad j = 1, 2, \ldots, m, \tag{44}$$

we obtain $P_{\sigma(j)} = 0$, hence it becomes a stable matrix. Since all matrices $P_{\sigma(i)}$ have the same eigenvalues, with this choice all matrices $P_{\sigma(i)}$ become stable, $i = 1, 2, \ldots, m$. Also note that the matrices $J_{\sigma(j)}$ also share the same eigenvalues. We can summarize these results as follows.

Theorem 2. *Let $\Sigma_m = \{x_1^*, x_2^*, \ldots, x_m^*\}$ be a period m orbit of (1) and let us define the related Jacobians J_i as given in (14). Consider the system given by (6),* (10), (11). *There exists a gain matrix K such that Σ_m is locally exponentially stable if and only if the matrix $J_{\sigma(1)}$ (and hence all $J_{\sigma(j)}$) does not have an eigenvalue $\lambda = 1$.*

Proof. Note that the local exponential stability is equivalent to the stability of the linearized system, see, e.g., [20]. The sufficiency follows from the analysis given above and from the fact that all matrices $J_{\sigma(j)}$ share the same eigenvalues, $i = 1, 2, \dots, m$. Conversely, assume that $J_{\sigma(1)}$ has an eigenvalue $\lambda = 1$, and let ϕ be the corresponding eigenvector, i.e., we have $J_{\sigma(1)}\phi = \phi$. Similar to the calculations made in (32), we obtain:

$$P_1\phi = \big[J_1 J_{\sigma(1)} + K(J_{\sigma(1)} - I)\big]\phi = J_1\phi, \tag{45}$$

$$\begin{aligned} P_2 P_1\phi = P_2 J_1\phi &= \big[J_2 J_{\sigma(2)} + K(J_{\sigma(2)} - I)\big] J_1\phi \\ &= \big[J_2 J_1 J_m \cdots J_2 + K(J_1 J_m \cdots J_2 - I)\big] J_1\phi \\ &= \big[J_2 J_1 J_{\sigma(1)} + K J_1(J_{\sigma(1)} - I)\big]\phi \\ &= J_2 J_1\phi. \end{aligned} \tag{46}$$

Similarly we obtain

$$P_j P_{j-1} \cdots P_2 P_1\phi = J_j J_{j-1} \cdots J_2 J_1\phi, \quad j = 1, 2, \dots, m. \tag{47}$$

Hence we have

$$P_{\sigma(1)}\phi = J_{\sigma(1)}\phi = \phi. \tag{48}$$

Therefore, $P_{\sigma(1)}$ has an eigenvalue $\lambda = 1$, hence is not stable, therefore for the original system Σ_m cannot be locally exponentially stable. □

Remark 3. By choosing $K = K_j$, where K_j is given by (44), $j = 1, 2, \dots, m$, we can stabilize Σ_m. Similar to Remark 2, there exist constants $\bar{\Delta}_j$, $j = 1, 2, \dots, m$ such that for any gain matrix K satisfying $\|K - K_j\| < \bar{\Delta}_j$, stabilization occurs.

Remark 4. Note that with the proposed scheme, only the periodic orbits Σ_m for which $J_{\sigma(1)}$ has at least one eigenvalue $\lambda = 1$ cannot be exponentially stabilized. A periodic orbit Σ_m is called hyperbolic if none of the eigenvalues of $J_{\sigma(1)}$ has unit magnitude, see, e.g., [21]. Hence any hyperbolic periodic orbit can be stabilized with the proposed scheme. On the other hand, for the non-hyperbolic case some eigenvalues of $J_{\sigma(1)}$ may have unit magnitude, and some of these orbits may be stabilized with the proposed scheme depending on the location of the eigenvalues. For classification purposes, we consider the following 2 cases:

(i) *Type* 1 *non-hyperbolic case*: In this case, at least one eigenvalue of $J_{\sigma(1)}$ has value 1. This is related to fold type bifurcation, see [18], or saddle-node type bifurcation, see, e.g., [19].

(ii) *Type* 2 *non-hyperbolic case*: In this case, $J_{\sigma(1)}$ does not have an eigenvalue at 1, but has some eigenvalues of the form $e^{\jmath\theta}$, $0 < \theta < \pi$, and/or at least an eigenvalue -1. The first case is related to a Hopf bifurcation, see, e.g., [19], and the second case is related to a flip bifurcation, see, e.g., [18,19].

By using the classification given above, we state that all hyperbolic periodic orbits as well as type 2 non-hyperbolic periodic orbits can be stabilized with the proposed method.

5. A simple implementation

Note that the SPDFC scheme given above achieves only local stabilization, i.e., it achieves stabilization only when the solutions of (1) are sufficiently close to the periodic orbit. Hence, from implementation point of view, it is reasonable to apply SPDFC only when the solutions are sufficiently close to Σ_m. Let $\epsilon_m > 0$ denote a constant related to the size of the domain of attraction of Σ_m. A reasonable implementation of SPDFC, which we will use in our simulations, is given as follows:

$$x(k+1) = f\big(x(k)\big) + u(k), \tag{49}$$

$$u(k) = \epsilon(k)\big(x(k) - x(k-m)\big), \tag{50}$$

$$\epsilon(k) = \begin{cases} K, & k\ \big(\text{mod } (m+1)\big) = 0\ \& \\ & d\big(x(k), \Sigma_m\big) < \epsilon_m, \\ 0, & \text{otherwise}, \end{cases} \tag{51}$$

where we compute $d_k(j)$ a little different that the one given in (3), see also (4). The reason is very simple: since T iterates of (1) starting from $x(k)$ are compared with Σ_T in $d_k(j)$, whereas to compute $u(k)$ we could only use the past iterates. For this reason, instead of (3), we modify $d_k(j)$ in this section as follows ($j = 1, 2, \dots, T$):

$$d_k(j) = \sqrt{\sum_{i=0}^{T-1} \left\| x(k-T+1+i) - x^*_{i+j} \right\|^2} \tag{52}$$

and compute $d(x(k), \Sigma_m)$ as given by (4). With this modification, we always compute the past T iterates of $x(k)$ with the circular permutations of the periodic points in Σ_m. Since the solutions of (49) are chaotic for $u = 0$, eventually the trajectories of the uncontrolled system will enter into the domain of attraction of Σ_m, i.e., $d(x(k), \Sigma_T) < \epsilon_m$ will be satisfied for some k, and hence afterwards the SPDFC given by (49)–(51) will be effective. Also, with this modification SPDFC will achieve stabilization for any initial condition in the domain of attraction of the chaotic attractor of (1). Obviously, for higher order periodic orbits, the time required till the trajectories enter into the domain of attraction of Σ_m will be larger.

6. Simulation results

In the simulations, we used the system given by (49)–(51) for various well-known chaotic maps. We will first consider the one-dimensional tent map given below:

$$f(x) = \begin{cases} \mu x, & 0 \leqslant x < 0.5, \\ \mu(1-x), & 0.5 \leqslant x \leqslant 1, \end{cases} \tag{53}$$

where $\mu = 1.9$. It is well known that this map has chaotic solutions and periodic orbits of all orders. Two true period 3 orbits of this map can be computed as $\Sigma_{3-} = \{0.872757, 0.241761, 0.459345\}$, $\Sigma_{3+} = \{0.846390, 0.291858, 0.554531\}$. For Σ_{3-}, we have $J_1 = -1.9$, $J_2 = J_3 = 1.9$, $J_{\sigma(1)} = -6.859$. Obviously, the condition in Theorem 2 is satisfied and this periodic orbit can be stabilized by using SPDFC. Note that by using the necessary and sufficient conditions given in [8], it can be shown that this orbit cannot be stabilized by using classical DFC. Since $J_2 = J_3$, and due to the scalar nature of Jacobians we have $J_{\sigma(1)} = J_{\sigma(2)} = J_{\sigma(3)}$, by using (44) we obtain $K_1 = 1.65823$, $K_2 = K_3 = -1.65823$, and by using the ideas given in Remarks 2 and 3, we see that Σ_{3-} can be exponentially stabilized when $1.65805 < K < 1.658421$, or $-1.68223 < K < -1.63423$. For this case, since the stabilization interval for $K_2 = K_3$ is larger, we choose $K = -1.65$, and by extensive numerical simulations we find that we have $\varepsilon_m = 0.1$. Our simulations show exponential stabilization for any $x(1) \in (0\ 1)$. We present a particular simulation result starting with $x(1) = 0.1$. The simulation results are shown in Fig. 1, where $d(x(k), \Sigma_{3-})$ vs. k and $u(k)$ vs. k are shown in Fig. 1(a) and (b). As can be seen, the trajectory converges to Σ_{3-} for $k \geqslant 200$. To show the asymptotic periodic behaviour, we show $x(k)$ vs. k for $980 \leqslant k \leqslant 1000$ and $x(k)$ vs. $x(k-3)$ for $k \geqslant 200$ in Fig. 1(c) and (d).

For Σ_{3+}, we have $J_1 = J_3 = -1.9$, $J_2 = 1.9$, $J_{\sigma(1)} = 6.859$, and since $J_{\sigma(1)} > 1$, this orbit cannot be stabilized by classical DFC [7,8]. Since $J_1 = J_3$, and due to the scalar nature of Jacobians we have $J_{\sigma(1)} = J_{\sigma(2)} = J_{\sigma(3)}$, by using (44) we obtain $K_1 = K_3 = 2.22428$, $K_2 = -2.22428$, and by using the ideas given in Remarks 2 and 3, we see that Σ_{3+} can be exponentially stabilized when $2.19278 < K < 2.25578$, or $-2.22453 < K < -2.22404$. Since the first stabilization interval is larger, in this case we choose $K = 2.2$, and by extensive numerical simulations we find that we have $\varepsilon_m = 0.1$. Our simulations show exponential stabilization for any $x(1) \in (0\ 1)$. We present a particular simulation result starting with $x(1) = 0.1$. The simulation results are shown in Fig. 2, where $d(x(k), \Sigma_{3+})$ vs. k and $u(k)$ vs. k are shown in Fig. 2(a) and (b). As can be seen, the trajectory converges to Σ_{3+} for $k \geqslant 200$. To show the asymptotic periodic behaviour, we show $x(k)$ vs. k for $980 \leqslant k \leqslant 1000$ and $x(k)$ vs. $x(k-3)$ for $k \geqslant 200$ in Fig. 2(c) and (d).

For 2-dimensional case, we choose the well-known lozi map given below:

$$x(k+1) = 1 + y(k) - a\left|x(k)\right|, \tag{54}$$

$$y(k+1) = bx(k), \tag{55}$$

where $a = 1.7$ and $b = 0.4$. It is well known that this system exhibits chaotic behaviour and has a large number of unstable periodic orbits. Let us denote $z = (x\ \ y)^T$. The system given above has a period 4 orbit $\Sigma_4 = \{z_1^*, z_2^*, z_3^*, z_4^*\}$ which is given as:

$$z_1^* = \begin{pmatrix} 0.112942579 \\ 0.153283946 \end{pmatrix},$$

$$z_2^* = \begin{pmatrix} 0.96128156 \\ 0.045177031 \end{pmatrix}, \tag{56}$$

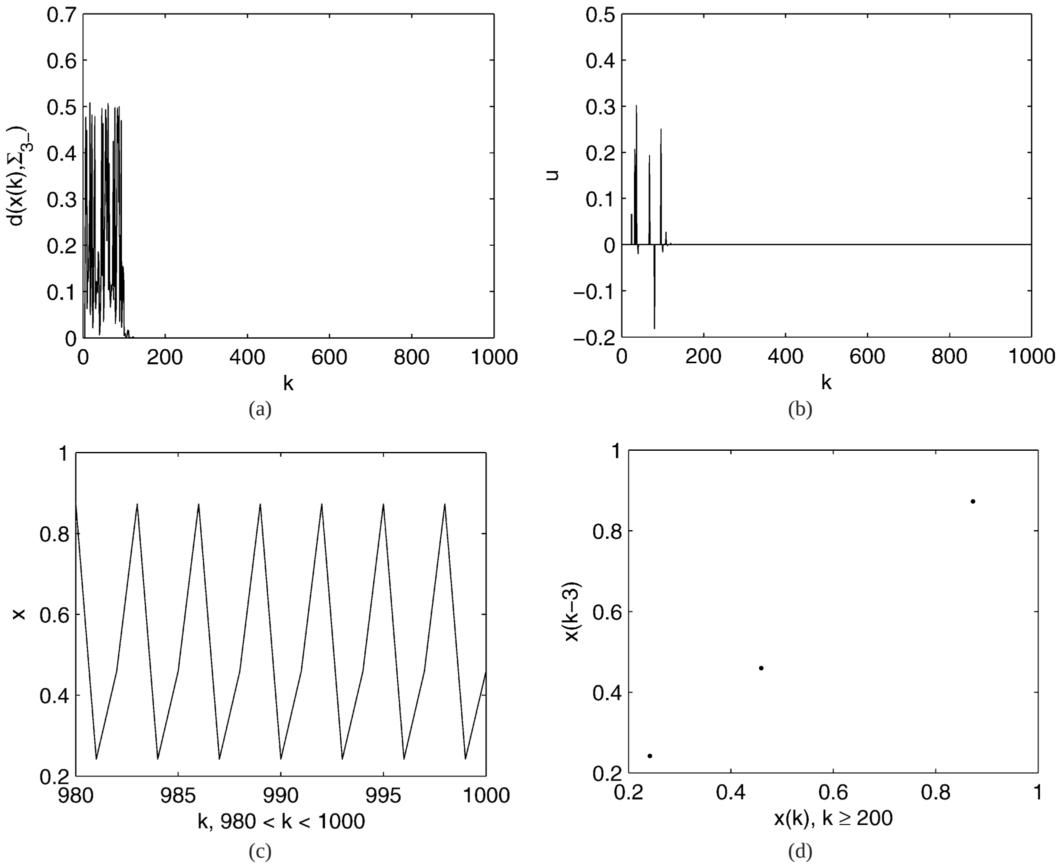

Fig. 1. SPDFC applied to tent map, (a) $d(x(k),\Sigma_{3-})$ vs. k, (b) $u(k)$ vs. k, (c) $x(k)$ vs. k for $980 \leqslant k \leqslant 1000$, (d) $x(k)$ vs. $x(k-3)$ for $k \geqslant 200$.

$$z_3^* = \begin{pmatrix} -0.589001622 \\ 0.384512624 \end{pmatrix},$$

$$z_4^* = \begin{pmatrix} 0.383209865 \\ -0.235600649 \end{pmatrix}. \tag{57}$$

For this system, by using (44), we find the following gain matrix:

$$K = K_1 = \begin{pmatrix} 1.54234408293496 & -0.77963500667940 \\ -0.31185400267176 & 0.15690517460282 \end{pmatrix}. \tag{58}$$

The remaining gains K_2, K_3, K_4 which also achieve exponential stabilization can be found by using (44), and the bounds on these gains can be found by using the ideas given in the Remarks 2 and 3. By extensive numerical simulations we find that we have $\varepsilon_m = 0.1$. Our simulations show exponential stabilization for any $x(1), y(1) \in (0\ 1)$. We present a particular simulation result starting with $x(1) = y(1) = 0.5$ in Fig. 3. The simulation results are shown in Fig. 3, where $d(x(k),\Sigma_4)$ vs. k, $u_1(k)$ and $u_2(k)$ vs. k are shown in Fig. 3(a)–(c), respectively. As can be seen, the trajectory converges to Σ_4 for $k \geqslant 600$. Finally, we show $x(k)$ vs. $y(k)$ in Fig. 3(d) for $k \geqslant 600$, which also confirms that the trajectory converges to Σ_4.

Remark 5. The input $u(k)$ given by (6) has the same dimension of $x(k)$. As noted by one of the reviewers, in some cases the dimension of u is required to be less than that of x. In such cases, (6) can be replaced by

$$x(k+1) = f\big(x(k)\big) + Bu(k), \tag{59}$$

where $B \in \mathbf{R}^{n\times q}$ and $q < n$. Therefore, $u(k) \in \mathbf{R}^{q\times n}$, hence its dimension is less than that of x. The analysis given above does not apply to this case directly, in

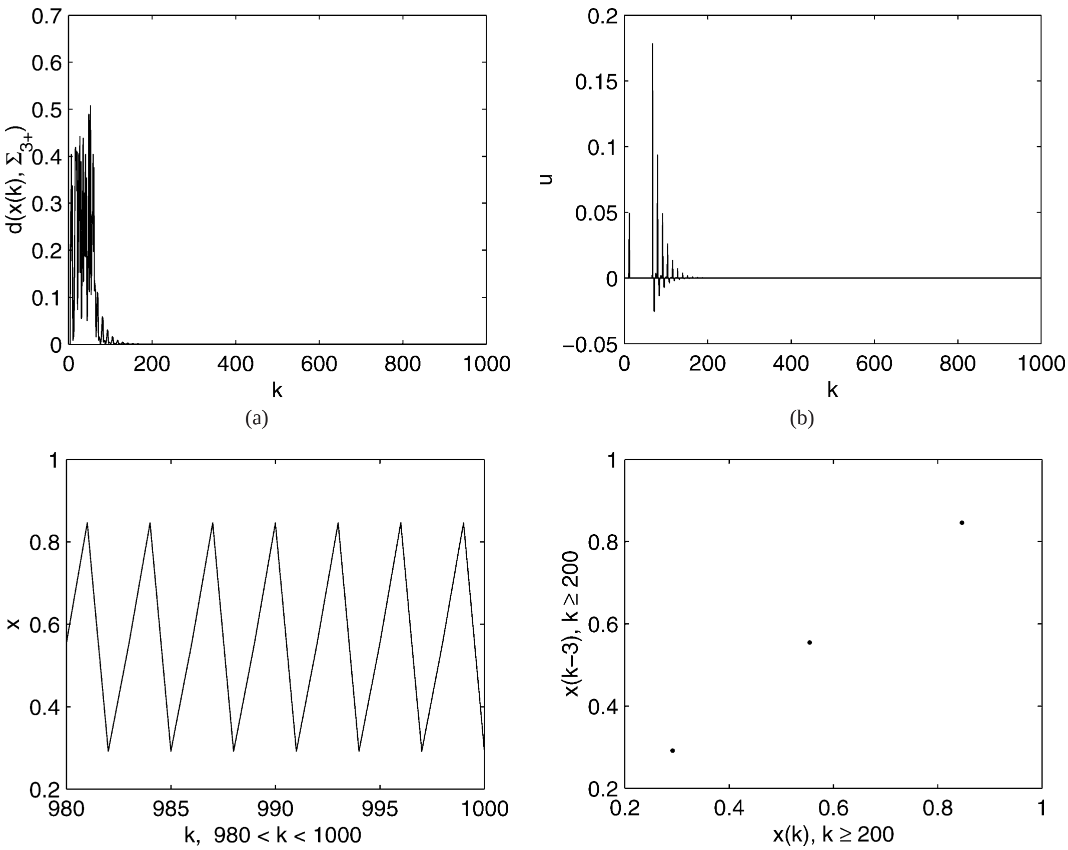

Fig. 2. SPDFC applied to tent map, (a) $d(x(k), \Sigma_{3+})$ vs. k, (b) $u(k)$ vs. k, (c) $x(k)$ vs. k for $980 \leqslant k \leqslant 1000$, (d) $x(k)$ vs. $x(k-3)$ for $k \geqslant 200$.

general. Note that in this case, by using (7)–(11), we may choose $\epsilon(k) = C^T$ where $C \in \mathbf{R}^{n\times q}$, hence we can write the gain matrix K given in (7) as $K = BC^T$. If B is not given, one approach might be to perturb the matrices K_j given by (44) so that $K = K_j + \Delta K$ has the form of BC^T. By Remark 3, if ΔK is sufficiently small, the analysis given above still holds. The applicability of this approach might be limited. One possible application of this approach is the stabilization of a saddle type Σ_m when m is large. In such cases, q eigenvalues of J_σ given in (44) will be unstable, whereas the remaining $n-q$ of them will be quite close to zero, hence the gain matrix K given by (44) will be quite close to a rank q matrix. Hence, by a small perturbation, i.e., by choosing $K = K_j + \Delta K$ with an appropriate ΔK, we may obtain a gain matrix with $\text{rank}(K) = q$, therefore we may express K as $K = BC^T$ with the dimensions as given above. By Remark 3, if ΔK is sufficiently small, then the stability analysis given above will be valid. As an example, consider the last simulation given above. Here, Σ_4 is of saddle type and the eigenvalues of J_σ are -8.0289 and -0.0032, hence we have $q = 1$ in this case. The eigenvalues of K_1 given by (58) are 1.6999 and -0.0007, hence K_1 is very close to a rank 1 matrix. Therefore by perturbing any entry of K_1 by an appropriately small amount, we may obtain a rank 1 gain matrix $\hat{K}$ which is very close to K_1. Let us choose $\hat{K}$ with entries $\hat{k}_{11} = k_{11}$, $\hat{k}_{12} = k_{12}$, $\hat{k}_{21} = k_{21}$ and $\hat{k}_{22} = 0.15763816916478$. Note that $\|\hat{K} - K_1\| = |\hat{k}_{22} - k_{22}| = 0.000732$, and $\hat{K}$ is a rank 1 matrix. Therefore we may write $K = BC^T$ with $B = (r\ 1)^T$, $C = (c_1\ c_2)$ with $r = k_{11}/k_{21} = -4.94572482546695$, $c_1 = k_{11}/r = -0.31185400267176$, and $c_2 = k_{12}/r =$

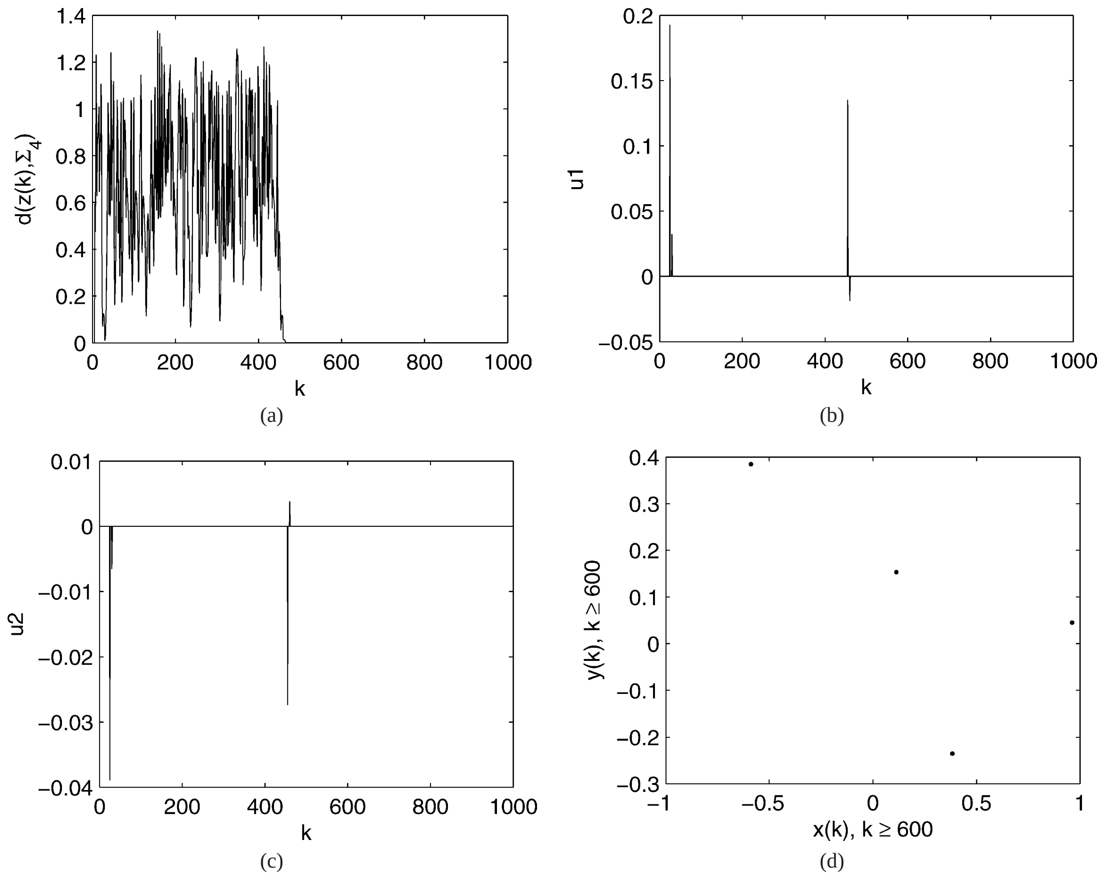

Fig. 3. SPDFC applied to Lozi map, multi input case, (a) $d(z(k), \Sigma_4)$ vs. k, (b) $u_1(k)$ vs. k, (c) $u_2(k)$ vs. k, (d) $x(k)$ vs. $y(k)$ for $k \geqslant 600$.

0.15763816916478. Note that in this case, we have (59) with B as given above, and $u(k) = c_1(x(k) - x(k-4)) + c_2(y(k) - y(k-4))$ when $k \pmod{5} = 0$, and $u(k) = 0$ otherwise. We simulated this case with $x(1) = y(1) = 0.5$, $\varepsilon_m = 0.1$, and the simulation results are shown in Fig. 4, where $d(x(k), \Sigma_4)$ vs. k, and $u(k)$ vs. k are shown in Fig. 4(a) and (c), respectively. As can be seen, the trajectory converges to Σ_4 for $k \geqslant 200$. We also show $x(k)$ for $980 \leqslant k \leqslant 995$ and $x(k)$ vs. $y(k)$ for $k \geqslant 200$, in Fig. 4(c) and (d), respectively, which also confirms that the trajectory converges to Σ_4.

On the other hand, if B is given, one may still try to apply the procedures given above but most probably the applicability will be limited. In such cases, as stated by one of the reviewers, the controllability of the linearized version of (59) around Σ_m should be taken into consideration, and most probably the stability analysis will be very complicated. Obviously this point is worth investigating and requires further research.

7. Conclusion

In this Letter we considered the stabilization of unstable periodic orbits of discrete time chaotic systems. We proposed a simple periodic delayed feedback scheme, which we called as Single Period DFC (SPDFC), and present some stability results. These results show that all hyperbolic periodic orbits as well as some non-hyperbolic periodic orbits can be stabilized with the proposed scheme. We also presented a scheme to choose the required gain matrix to achieve stabilization, see Remark 1.

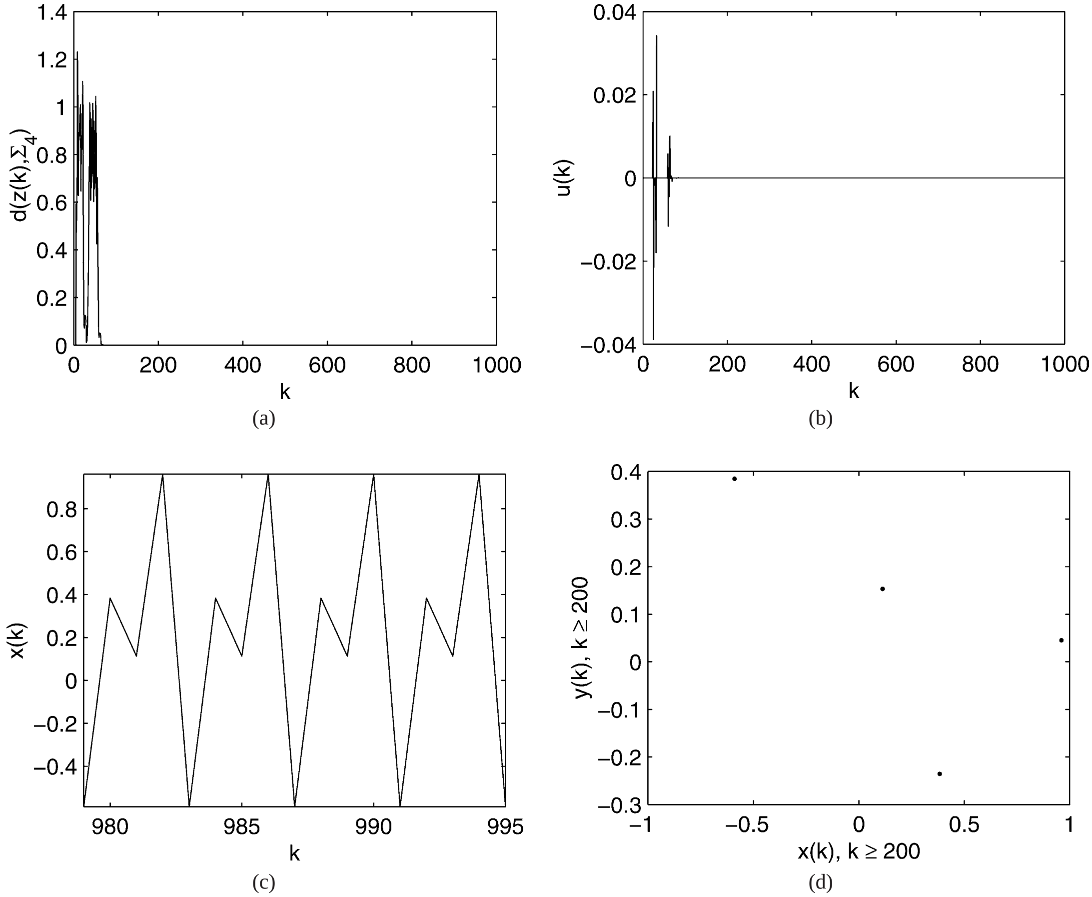

Fig. 4. SPDFC applied to Lozi map, single input case (a) $d(z(k), \Sigma_4)$ vs. k, (b) $u(k)$ vs. k, (c) $x(k)$ for $980 \leqslant k \leqslant 995$, (d) $x(k)$ vs. $y(k)$ for $k \geqslant 200$.

The proposed method may not achieve stabilization only when the given periodic orbit is of type 1 non-hyperbolic orbit, see Remark 3. This type of periodic orbits may occur due to fold bifurcation [18], or saddle-node bifurcations [19]. An interesting open problem may be to modify the proposed scheme to achieve stabilization for the case mentioned above as well.

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